

AN IDENTITY OF THE SYMMETRY FOR THE FROBENIUS-EULER POLYNOMIALS ASSOCIATED WITH THE FERMIONIC p -ADIC INVARIANT q -INTEGRALS ON \mathbb{Z}_p

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ABSTRACT. The main purpose of this paper is to prove an identity of symmetry for the Frobenius-Euler polynomials. It turns out that the recurrence relation and multiplication theorem for the Frobenius-Euler polynomials which discussed in [K. Shiratani, S. Yamamoto, On a p -adic interpolation function for the Euler numbers and its derivatives, Memo. Fac. Sci. Kyushu University Ser.A, 39(1985), 113-125]. Finally we investigate several further interesting properties of symmetry for the fermionic p -adic invariant q -integral on \mathbb{Z}_p associated with the Frobenius-Euler polynomials and numbers.

§1. Introduction

The n -th Frobenius-Euler numbers $H_n(q)$ and the n -th Frobenius-Euler polynomials $H_n(q, x)$ attached to an algebraic number $q (\neq 1)$ may be defined by the exponential generating functions

$$(1) \quad \sum_{n=1}^{\infty} H_n(q) \frac{t^n}{n!} = \frac{1-q}{e^t - q}, \text{ see [6,7],}$$

$$\sum_{n=0}^{\infty} H_n(q, x) \frac{t^n}{n!} = \frac{1-q}{e^t - q} e^{xt}.$$

It is easy to show that $H_n(q, x) = \sum_{l=0}^n \binom{n}{l} x^{n-l} H_l(q)$. Let p be a fixed prime. Throughout this paper \mathbb{Z}_p , \mathbb{Q}_p , \mathbb{C} , and \mathbb{C}_p will, respectively, denote the ring of p -adic rational integers, the field of p -adic rational numbers, the complex number field, and the completion of algebraic closure of \mathbb{Q}_p . When one talks of q -extension, q is variously considered as an indeterminate, a complex $q \in \mathbb{C}$, or a p -adic number $q \in \mathbb{C}_p$, see [9-22]. If $q \in \mathbb{C}$, then we assume $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume

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$|1 - q|_p < 1$. For $x \in \mathbb{Q}_p$, we use the notation $[x]_q = \frac{1-q^x}{1-q}$, and $[x]_{-q} = \frac{1-(-q)^x}{1+q}$, see [5-6]. The normalized valuation in \mathbb{C}_p is denoted by $|\cdot|_p$ with $|p|_p = \frac{1}{p}$. We say that f is a uniformly differentiable function at a point $a \in \mathbb{Z}_p$ and denote this property by $f \in UD(\mathbb{Z}_p)$, if the difference quotients $F_f(x, y) = \frac{f(x) - f(y)}{x - y}$ have a limit $l = f'(a)$ as $(x, y) \rightarrow (a, a)$. For $f \in UD(\mathbb{Z}_p)$, let us start with the expression

$$\frac{1}{[p^N]_q} \sum_{0 \leq j < p^N} q^j f(j) = \sum_{0 \leq j < p^N} f(j) \mu_q(j + p^N \mathbb{Z}_p),$$

representing a q -analogue of Riemann sums for f , see [5, 6]. The integral of f on \mathbb{Z}_p will be defined as limit ($n \rightarrow \infty$) of those sums, when it exists. The q -deformed bosonic p -adic integral of the function $f \in UD(\mathbb{Z}_p)$ is defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \rightarrow \infty} \frac{1}{[dp^N]_q} \sum_{0 \leq x < dp^N} f(x) q^x, \text{ see [5].}$$

Thus, we note that

$$qI_q(f_1) = I_q(f) + (q - 1)f(0) + \frac{q - 1}{\log q} f'(0),$$

where $f_1(x) = f(x + 1)$, $f'(0) = \frac{df(0)}{dx}$.

The fermionic p -adic invariant q -integral on \mathbb{Z}_p is defined as

$$(2) \quad I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x) (-q)^x, \text{ see [5].}$$

In [8], H.J.H. Tuentier provided a generalization of the Bernoulli number recurrence

$$B_m = \frac{1}{a(1 - a^m)} \sum_{j=0}^{m-1} a^j \binom{m}{j} B_j \sum_{i=0}^{a-1} i^{m-j}, \text{ see [2, 3, 4],}$$

where $a, m \in \mathbb{Z}$ with $a > 1$, $m \geq 1$, attributed to E.Y. Deeba and D.M. Rodriguez[2] and to I. Gessel[3]. Define $S_m(k) = 0^m + 1^m + \cdots + k^m$, where $a, m \in \mathbb{Z}$, with $a \geq 0$ and $m \geq 0$. H.J.H. Tuentier proved that the quantity

$$\sum_{j=0}^m \binom{m}{j} a^{j-1} B_j b^{m-j} S_{m-j}(a - 1), \text{ see [8],}$$

is symmetric in a and b , provided $a, b, m \in \mathbb{Z}$, with $a > 0, b > 0$ and $m \geq 0$. In this paper we prove an identity of symmetry for the Frobenius-Euler polynomials. It

turns out that the recurrence relation and multiplication theorem for the Frobenius-Euler polynomials which discussed in [7]. Finally we investigate the several further interesting properties of the symmetry for the fermionic p -adic invariant q -integral on \mathbb{Z}_p associated with the Frobenius-Euler polynomials and numbers.

§2. An identity of symmetry for the Frobenius-Euler polynomials

From (2) we can derive

$$(3) \quad qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0), \text{ where } f_1(x) = f(x+1).$$

By continuing this process, we see that

$$q^n I_{-q}(f_n) + (-1)^{n-1} I_{-q}(f) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l f(l), \text{ where } f_n(x) = f(x+n).$$

When n is an odd positive integer, we obtain

$$(4) \quad q^n I_{-q}(f_n) + I_{-q}(f) = [2]_q \sum_{l=0}^{n-1} (-1)^l f(l) q^l.$$

If $n \in \mathbb{N}$ with $n \equiv 0 \pmod{2}$, then we have

$$(5) \quad q^n I_{-q}(f_n) - I_{-q}(f) = [2]_q \sum_{l=0}^{n-1} (-1)^{l-1} f(l) q^l.$$

From (1) and (3) we derive

$$(6) \quad \int_{\mathbb{Z}_p} e^{xt} d\mu_{-q}(x) = \frac{1 - (-q)^{-1}}{e^t - (-q)^{-1}} = \sum_{n=0}^{\infty} H_n(-q^{-1}) \frac{t^n}{n!}.$$

Thus, we note that

$$\int_{\mathbb{Z}_p} x^n d\mu_{-q}(x) = H_n(-q^{-1}), \text{ and } \int_{\mathbb{Z}_p} (y+x)^n d\mu_{-q}(x) = H_n(-q^{-1}, y).$$

Let $n \in \mathbb{N}$ with $n \equiv 1 \pmod{2}$. Then we obtain

$$[2]_q \sum_{l=0}^{n-1} (-1)^l q^l l^m = q^n H_m(-q^{-1}, n) + H_m(-q^{-1}).$$

For $n \in \mathbb{N}$ with $n \equiv 0 \pmod{2}$, we have

$$q^n H_m(-q^{-1}, n) - H_m(-q^{-1}) = [2]_q \sum_{l=0}^{n-1} (-1)^{l-1} q^l l^m.$$

By substituting $f(x) = e^{xt}$ into (4), we can easily see that

$$(7) \quad \int_{\mathbb{Z}_p} q^n e^{(x+n)t} d\mu_{-q}(x) + \int_{\mathbb{Z}_p} e^{xt} d\mu_{-q}(x) = [2]_q \frac{q^n e^{nt} + 1}{qe^t + 1} = [2]_q \sum_{l=0}^{n-1} (-1)^l q^l e^{lt}.$$

Let $S_{k,q}(n) = \sum_{l=0}^n (-1)^l l^k q^l$. Then $S_{k,q}(n)$ is called by the alternating sums of powers of consecutive q -integers. From the definition of the fermionic p -adic invariant q -integral on \mathbb{Z}_p , we can derive

$$(8) \quad \int_{\mathbb{Z}_p} q^n e^{(x+n)t} d\mu_{-q}(x) + \int_{\mathbb{Z}_p} e^{xt} d\mu_{-q}(x) = \frac{[2]_q \int_{\mathbb{Z}_p} e^{xt} d\mu_{-q}(x)}{\int_{\mathbb{Z}_p} e^{nxt} q^{(n-1)x} d\mu_{-q}(x)}.$$

By (8), we easily see that

$$\int_{\mathbb{Z}_p} q^{(n-1)x} e^{nxt} d\mu_{-q}(x) = \frac{1+q}{q^n e^{nt} + 1}.$$

Let $w_1, w_2 \in \mathbb{N}$ be odd. By using double fermionic p -adic invariant q -integral on \mathbb{Z}_p , we obtain

$$\frac{\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} e^{(w_1 x_1 + w_2 x_2)t} d\mu_{-q}(x_1) d\mu_{-q}(x_2)}{\int_{\mathbb{Z}_p} e^{w_1 w_2 x t} q^{(w_1 w_2 - 1)x} d\mu_{-q}(x)} = \frac{[2]_q (q^{w_1 w_2} e^{w_1 w_2 t} + 1)}{(qe^{w_1 t} + 1)(qe^{w_2 t} + 1)}.$$

Now we also consider the following fermionic p -adic invariant q -integral on \mathbb{Z}_p associated with Frobenius-Euler polynomials.

$$(9) \quad \begin{aligned} I &= \frac{\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} e^{(w_1 x_1 + w_2 x_2 + w_1 w_2 x)t} d\mu_{-q}(x_1) d\mu_{-q}(x_2)}{\int_{\mathbb{Z}_p} e^{w_1 w_2 x t} q^{(w_1 w_2 - 1)x} d\mu_{-q}(x)} \\ &= \frac{[2]_q e^{w_1 w_2 x t} (q^{w_1 w_2} e^{w_1 w_2 t} + 1)}{(qe^{w_1 t} + 1)(qe^{w_2 t} + 1)}. \end{aligned}$$

From (9) and (8), we can derive

$$\begin{aligned}
(10) \quad \frac{[2]_q \int_{\mathbb{Z}_p} e^{xt} d\mu_{-q}(x)}{\int_{\mathbb{Z}_p} e^{w_1 xt} q^{(w_1-1)x} d\mu_{-q}(x)} &= [2]_q \sum_{l=0}^{w_1-1} (-1)^l q^l e^{lt} = \sum_{k=0}^{\infty} \left([2]_q \sum_{l=0}^{w_1-1} (-1)^l q^l l^k \right) \frac{t^k}{k!} \\
&= \sum_{k=0}^{\infty} [2]_q S_{k,q}(w_1-1) \frac{t^k}{k!}.
\end{aligned}$$

By (9) and (10), we easily see that

$$\begin{aligned}
(11) \quad I &= \left(\frac{1}{[2]_q} \int_{\mathbb{Z}_p} e^{w_1(x_1+w_2x)t} d\mu_{-q}(x) \right) \left(\frac{[2]_q \int_{\mathbb{Z}_p} e^{w_2x_2t} d\mu_{-q}(x_2)}{\int_{\mathbb{Z}_p} e^{w_1w_2xt} q^{(w_1w_2-1)x} d\mu_{-q}(x)} \right) \\
&= \left(\frac{1}{[2]_q} \sum_{i=0}^{\infty} H_i(-q^{-1}, w_2x) \frac{w_1^i}{i!} t^i \right) \left([2]_q \sum_{l=0}^{\infty} S_{l,qw_2}(w_1-1) \frac{w_2^l}{l!} t^l \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{i=0}^n \binom{n}{i} H_i(-q^{-1}, w_2x) S_{n-i,qw_2}(w_1-1) w_1^i w_2^{n-i} \right) \frac{t^n}{n!},
\end{aligned}$$

where $H_n(-q^{-1}, x)$ are the n -th Frobenius-Euler polynomials.

On the other hand,

$$\begin{aligned}
(12) \quad I &= \left(\frac{1}{[2]_q} \int_{\mathbb{Z}_p} e^{w_2(x_2+w_1x)t} d\mu_{-q}(x_2) \right) \left(\frac{[2]_q \int_{\mathbb{Z}_p} e^{w_1x_1t} d\mu_{-q}(x_1)}{\int_{\mathbb{Z}_p} e^{w_1w_2xt} q^{(w_1w_2-1)x} d\mu_{-q}(x)} \right) \\
&= \frac{1}{[2]_q} \left(\sum_{i=0}^{\infty} H_i(-q^{-1}, w_1x) \frac{w_2^i}{i!} t^i \right) \left([2]_q \sum_{l=0}^{\infty} S_{l,qw_1}(w_2-1) \frac{w_1^l}{l!} t^l \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{i=0}^n \binom{n}{i} H_i(-q^{-1}, w_1x) S_{n-i,qw_1}(w_2-1) w_2^i w_1^{n-i} \right) \frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficients on the both sides of (11) and (12), we obtain the following theorem.

Theorem 1. *Let $w_1, w_2 \in \mathbb{N}$ be odd and let $n \geq 0$ with $n \equiv 1 \pmod{2}$. Then we have*

$$\begin{aligned}
(13) \quad &\sum_{i=0}^n \binom{n}{i} H_i(-q^{-1}, w_2x) S_{n-i,qw_2}(w_1-1) w_1^i w_2^{n-i} \\
&= \sum_{i=0}^n \binom{n}{i} H_i(-q^{-1}, w_1x) S_{n-i,qw_1}(w_2-1) w_2^i w_1^{n-i},
\end{aligned}$$

where $H_n(q, x)$ are the n -th Frobenius-Euler polynomials.

Setting $x = 0$ in (13), we obtain the following corollary.

Corollary 2. *Let $w_1, w_2 \in \mathbb{N}$ be odd and let $n \in \mathbb{Z}_+$ be an odd. Then we have*

$$\begin{aligned} & \sum_{i=0}^n \binom{n}{i} H_i(-q^{-1}) S_{n-i, q^{w_2}}(w_1 - 1) w_1^i w_2^{n-i} \\ &= \sum_{i=0}^n \binom{n}{i} H_i(-q^{-1}) S_{n-i, q^{w_1}}(w_2 - 1) w_2^i w_1^{n-i}, \end{aligned}$$

where $H_i(-q^{-1})$ are the n -th Frobenius-Euler numbers.

If we take $w_2 = 1$ in (13), then we have

$$(14) \quad H_n(-q^{-1}, w_1 x) = \sum_{i=0}^n \binom{n}{i} H_i(-q^{-1}, x) S_{n-i, q}(w_1 - 1) w_1^i.$$

Setting $x = 0$ in (14), we obtain the following corollary.

Corollary 3. *Let $w_1 (> 1)$ be an odd integer and let $n \in \mathbb{Z}_+$ with $n \equiv 1 \pmod{2}$. Then we have*

$$H_n(-q^{-1}) = \frac{1}{1 - w_1^n} \sum_{i=0}^{n-1} \binom{n}{i} H_i(-q^{-1}) S_{n-i, q}(w_1 - 1) w_1^i.$$

From (7) and (8), we derive

$$\begin{aligned} (15) \quad I &= \left(\frac{e^{w_1 w_2 x t}}{[2]_q} \int_{\mathbb{Z}_p} e^{w_1 x_1 t} d\mu_{-q}(x_1) \right) \left(\frac{[2]_q \int_{\mathbb{Z}_p} e^{w_2 x_2 t} d\mu_{-q}(x_2)}{\int_{\mathbb{Z}_p} e^{w_1 w_2 x t} q^{(w_1 w_2 - 1)x} d\mu_{-q}(x)} \right) \\ &= \left(\frac{e^{w_1 w_2 x t}}{[2]_q} \int_{\mathbb{Z}_p} e^{w_1 x_1 t} d\mu_{-q}(x_1) \right) \left([2]_q \sum_{l=0}^{w_1-1} (-1)^l q^{w_2 l} e^{w_2 l t} \right) \\ &= \sum_{l=0}^{w_1-1} (-1)^l q^{w_2 l} \int_{\mathbb{Z}_p} e^{(x_1 + w_2 x + (\frac{w_2}{w_1})l)t w_1} d\mu_{-q}(x_1) \\ &= \sum_{n=0}^{\infty} \left(w_1^n \sum_{l=0}^{w_1-1} (-1)^l q^{w_2 l} H_n(-q^{-1}, w_2 x + \frac{w_2}{w_1} l) \right) \frac{t^n}{n!}. \end{aligned}$$

On the other hand,

$$\begin{aligned}
(16) \quad I &= \left(\frac{e^{w_1 w_2 x t}}{[2]_q} \int_{\mathbb{Z}_p} e^{w_2 x_2 t} d\mu_{-q}(x_2) \right) \left(\frac{[2]_q \int_{\mathbb{Z}_p} e^{w_1 x_1 t} d\mu_{-q}(x_1)}{\int_{\mathbb{Z}_p} e^{w_1 w_2 x t} q^{(w_1 w_2 - 1)x} d\mu_{-q}(x)} \right) \\
&= \left(\frac{1}{[2]_q} \int_{\mathbb{Z}_p} e^{w_2 x_2 t} d\mu_{-q}(x_2) \right) \left([2]_q \sum_{l=0}^{w_2-1} (-1)^l q^{w_1 l} e^{(w_1 l + w_1 w_2 x) t} \right) \\
&= \sum_{l=0}^{w_2-1} (-1)^l q^{w_1 l} \int_{\mathbb{Z}_p} e^{(x_2 + w_1 x + \frac{w_1}{w_2} l) t w_2} d\mu_{-q}(x_2) \\
&= \sum_{n=0}^{\infty} \left(w_2^n \sum_{l=0}^{w_2-1} (-1)^l q^{w_1 l} H_n(-q^{-1}, w_1 x + \frac{w_1}{w_2} l) \right) \frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficients on the both sides of 915) and (160), we obtain the following theorem.

Theorem 4. *Let $w_1, w_2 \in \mathbb{N}$ be odd and let $n \in \mathbb{Z}_+$ with $n \equiv 1 \pmod{2}$. Then we have*

$$w_1^n \sum_{l=0}^{w_1-1} (-1)^l q^{w_2 l} H_n(-q^{-1}, w_2 x + \frac{w_2}{w_1} l) = w_2^n \sum_{l=0}^{w_2-1} (-1)^l q^{w_1 l} H_n(-q^{-1}, w_1 x + \frac{w_1}{w_2} l).$$

Setting $w_2 = 1$ in Theorem 4, we get the multiplication theorem for the Frobenius-Euler polynomials as follows:

$$H_n(-q^{-1}, w_1 x) = w_1^n \sum_{l=0}^{w_1-1} (-1)^l q^l H_n(-q^{-1}, x + \frac{l}{w_1}).$$

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